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On nonlinearity of p-brane dynamics

A. A. Zheltukhin [a,b,c*]

[a] Kharkov Institute of Physics and Technology,
1, Akademicheskaya St., Kharkov, 61108, Ukraine
[b] Physics Department, Stockholm University, AlbaNova,
106 91, Stockholm, Sweden
[c] NORDITA,
Roslagstullsbacken 23, 106 91 Stockholm, Sweden

Abstract

Nonlinear equations of p-branes in $D = (2p+1)$-dimensional Minkowski space are discussed. Presented are new exact solutions for a set of spinning p-branes with $p = 2, 3, ..., (D-1)/2$ and the Abelian symmetries $U(1) \times U(1) \times \ldots \times U(1)$ of their shapes.

1 Introduction

Membranes and p-branes play an important role in M/string theory [1], but the construction of general solutions for their classical equations meets serious problems caused by their nonlinearity [2-13]. This preserves the necessity of the search and classification of various particular solutions of the brane equations. Spinning membranes ($p = 2$) with spherical or toroidal topology embedded in flat backgrounds with or without toroidal compactifications, as well as in $AdS_p \times S^q$ space-times, form one of the sectors the equations for which were considered in [14], [15], [16], [18], [17]. These interesting results were generalized to the case of 3-branes ($p = 3$) with more general symmetries $SU(n) \times SU(m) \times SU(k)$ by complexifying the target space [19]

*e-mail: aaz@physto.se

and proving the radial stability of these configurations. Still it is important to find exact solutions of the brane equations in the explicit form, as well as to study spinning branes with any p.

Here we partially touch this problem and find exact solutions for the set of closed spinning $U(1)^p$-invariant p-branes with the topology of p-torus embedded in $D=(2p+1)$-dimensional Minkowski space. Our construction generalizes the $U(1)$ -invariant membrane anzats proposed by Hoppe in [20] and its exact solutions for $D=5$ found in [21]. We construct the new anzats starting from the exactly solvable one [22] and substitute p propagating polar angle coordinates instead of p propogating radial coordinates describing $U(1)^p$-invariant collapsing p-branes. For the case of the $U(1)^p$-invariant noncompact p-branes without boundaries our anzats describes spinning branes with the topology of p-dimensional hyperplanes. The constructed exact solutions for the case $p=5$ describe $U(1)^5$-invariant spinning 5-branes of M/string theory in D=11 space-time.

2 P-brane equations

The Dirac action for a p-brane without boundaries is defined by the integral

$$S = T\int \sqrt{|G|}d^{p+1}\xi,$$

in the dimensionless worldvolume parameters ξ^α $(\alpha = 0, \ldots, p)$. The components $x^m = (t, \vec{x})$ of the brane world vector in the D-dimensional Minkowski space with the signature $\eta_{mn} = (+, -, \ldots, -)$ have the dimension of length, and the dimension of tension T is $L^{-(p+1)}$. The induced metric $G_{\alpha\beta} := \partial_\alpha x_m \partial_\beta x^m$ is presented in S by its determinant G.

After splitting the parameters $\xi^\alpha := (\tau, \sigma^r)$ the Euler-Lagrange equations and $(p+1)$ primary constraints generated by S take the form

$$\partial_\tau \mathcal{P}^m = -T\partial_r(\sqrt{|G|}G^{r\alpha}\partial_\alpha x^m), \quad \mathcal{P}^m = T\sqrt{|G|}G^{\tau\beta}\partial_\beta x^m, \tag{1}$$

$$\tilde{T}_r := \mathcal{P}^m \partial_r x_m \approx 0, \quad \tilde{U} := \mathcal{P}^m \mathcal{P}_m - T^2|\det G_{rs}| \approx 0, \tag{2}$$

where $\mathcal{P}^m$ is the energy-momentum density of the brane.

It is convenient to use the orthogonal gauge simplifying the metric $G_{\alpha\beta}$

$$L\tau = x^0 \equiv t, \quad G_{\tau r} = -L(\dot{\vec{x}} \cdot \partial_r \vec{x}) = 0, \tag{3}$$

$$g_{rs} := \partial_r \vec{x} \cdot \partial_s \vec{x}, \quad G_{\alpha\beta} = \begin{pmatrix} L^2(1-\dot{\vec{x}}^2) & 0 \\ 0 & -g_{rs} \end{pmatrix}$$

with $\dot{\vec{x}} := \partial_t \vec{x} = L^{-1}\partial_\tau \vec{x}$. The solution of the constraint $\tilde{U}$ (2) takes the form

$$\mathcal{P}_0 = \sqrt{\vec{\mathcal{P}}^2 + T^2|g|}, \qquad g = \det(g_{rs}) \tag{4}$$

and becomes the Hamiltonian density $\mathcal{H}_0$ of the p-brane since $\dot{\mathcal{P}}_0 = 0$ in view of Eq. (1). Using the definition of $\mathcal{P}_0$ (1) and $G^{\tau\tau} = 1/L^2(1-\dot{\vec{x}}^2)$ we find the expression of $\mathcal{P}_0$ as a function of the p-brane velocity $\dot{\vec{x}}$

$$\mathcal{P}_0 := TL\sqrt{|detG|}G^{\tau\tau} = T\sqrt{\frac{|g|}{1-\dot{\vec{x}}^2}}\ . \tag{5}$$

Taking into account this expression for $\mathcal{P}_0$ and the definition (1) one can present $\vec{\mathcal{P}}$ and its evolution equation (1) as

$$\vec{\mathcal{P}} = \mathcal{P}_0\dot{\vec{x}}, \qquad \dot{\vec{\mathcal{P}}} = T^2\partial_r\left(\frac{|g|}{\mathcal{P}_0}g^{rs}\partial_s\vec{x}\right). \tag{6}$$

Then Eqs. (6) yield the second-order PDE for $\vec{x}$

$$\ddot{\vec{x}} = \frac{T}{\mathcal{P}_0}\partial_r\left(\frac{T}{\mathcal{P}_0}|g|g^{rs}\partial_s\vec{x}\right). \tag{7}$$

These equations may be presented in the canonical Hamiltonian form

$$\dot{\vec{x}} = \{H_0, \vec{x}\}, \qquad \dot{\vec{\mathcal{P}}} = \{H_0, \vec{\mathcal{P}}\}, \qquad \{\mathcal{P}_i(\sigma), x_j(\tilde{\sigma})\} = \delta_{ij}\delta^{(p)}(\sigma^r - \tilde{\sigma}^r)$$

using the integrated Hamiltonian density (4) $\mathcal{H}_0 (= \mathcal{P}_0)$

$$H_0 = \int d^p\sigma\sqrt{\vec{\mathcal{P}}^2 + T^2|g|}. \tag{8}$$

The presence of square root in (8) points to the presence of the known residual symmetry preserving the orthonormal gauge (3)

$$\tilde{t} = t, \qquad \tilde{\sigma}^r = f^r(\sigma^s) \tag{9}$$

and generated by the constraints $\tilde{T}_r$ (2) reduced to the form

$$T_r := \vec{\mathcal{P}}\partial_r\vec{x} = 0 \quad \Leftrightarrow \quad \dot{\vec{x}}\partial_r\vec{x} = 0, \quad (r = 1, 2, \ldots, p). \tag{10}$$

The freedom allows to impose p additional time-independent conditions on $\vec{x}$ and its space-like derivatives. The above description is valid for any space-time and brane worldvolume dimensions (D, p) with $p < D$.

3 $\mathbf{U(1)\times U(1)\times \ldots \times U(1)}$ spinning p-branes

Here we continue studying p-branes which evolve in $D=(2p+1)$-dimensional Minkowski space-time starting from the general representation for the $2p$-dimensional p-brane Euclidean vector $\vec{x}$ by p pairs of its "polar" coordinates

$$\vec{x}^T(t,\sigma^r)=(q_1\cos\theta_1, q_1\sin\theta_1,\ldots,q_p\cos\theta_p,q_p\sin\theta_p). \tag{11}$$

The polar brane coordinates $q_a=q_a(t,\sigma^r)$ with $a=1,\ldots,p$ and $\theta_a=\theta_a(t,\sigma^r)$ certainly depend on all the parameters $(t,\sigma^1,...,\sigma^p)$ of the p-brane worldvolume. However, such a dependence obstructs exact solvability of the brane equations (7). The separation of the time and σ^r variables, fixed by the choice $q_a=q_a(t)$ and $\theta_a=\theta_a(\sigma^r)$, transforms the representation (11) to an exactly solvable anzats describing contracted toroidal p-branes [22].

Here we consider an alternative anzats originating in (11), but with the "polar" angles $\theta_a=\theta_a(t)$ independent of σ^r and the time-dependent "radial" coordinates $q_a=q_a(\sigma^r)$

$$\vec{x}^T=(q_1\cos\theta_1,q_1\sin\theta_1,q_2\cos\theta_2,q_2\sin\theta_2,\ldots,q_p\cos\theta_p,q_p\sin\theta_p), \tag{12}$$
$$q_a=q_a(\sigma^r),\ \ \theta_a=\theta_a(t).$$

We show that anzats (12) is also exactly solvable and describes either spinning toroidal p-branes or spinning p-dimensional hyperplanes.

The brane space vector (12) lies in the $2p$-dimensional Euclidean subspace of the $(2p+1)$-dimensional Minkowski space and automatically satisfies to the orthogonality constraints (10): $\dot{\vec{x}}\partial_r\vec{x}=0$. At any fixed moment of time the world vector $\vec{x}^T$ (12) is produced from $\vec{x_0}^T=(q_1,0,q_2,0,\ldots,q_p,0)$ by rotations of the diagonal subgroup $U(1)^p\in SO(2p)$, parametrized by the angles $\theta_a(t)$ describing rotations in the planes x_1x_2, x_3x_4 ,...,$x_{2p-1}x_{2p}$. Each of these $U(1)$ symmetries is locally isomorphic to one of the $O(2)$ subgroups of the $SO(2p)$ group of the Euclidean rotations. Thus, the p-dimensional hypersurface Σ_p of $U(1)^p$-invariant p-brane has the Abelian group $U(1)\times U(1)\times\ldots\times U(1)\equiv U(1)^p$ as its isometry containing p Killing vectors. The Abelian character of the group $U(1)^p$ supposes the existence of a local parametrization of the p-brane hypersurface Σ_p with the metric tensor g_{rs} independent of σ^r. Actually, the metric $G_{\alpha\beta}$ of the $(p+1)$-dimensional worldvolume Σ_{p+1} created by (12) has the form similar to (3) with the non-zero components

$$G_{tt}=1-\sum_{a=1}^{p}(q_a\dot\theta_a)^2,\quad g_{rs}=\sum_{a=1}^{p}q_{a,r}q_{a,s}\equiv\mathbf{q}_{,r}\mathbf{q}_{,s},\quad \mathbf{q}:=(q_1,..,q_p), \tag{13}$$

where $\dot{\theta}_a \equiv \partial_t \theta_a, \quad q_{a,r} \equiv \partial_r q_a$ and yields the following interval ds^2 on Σ_{p+1}

$$ds^2_{p+1} = (1 - \sum_{a=1}^{p} (q_a \dot{\theta}_a)^2) dt^2 - \sum_{a=1}^{p} dq_a dq_a. \tag{14}$$

This representation shows that the change of σ^r parametrizing Σ_p by the new local coordinates $q_a(\sigma^r)$ makes the induced metric on Σ_p independent of σ^r. In the new parametrization the above mentioned Killing vector fields on Σ_p take the form of the derivatives $\frac{\partial}{\partial q_a}$.

This shows that the closed hypersurface Σ_p presented by (12) is a flat manifold with its shape isomorphic to a flat p-dimensional torus $S^1 \times S^1 \times \ldots \times S^1$ with p multipliers S^1. The next step is to show that anzats (12) is in fact an exact solution of the nonlinear chain (7).

4 Solutions of spinning brane equations

The substitution of anzats (12) into Eqs.(7) reduces these $2p$ nonlinear PDEs for the $\vec{x}$ components to p equations for the p components of $\mathbf{q}(\sigma^r)$

$$\dot{\theta}_a^2 q_a + \frac{T}{\mathcal{P}_0} \partial_r \left(\frac{T}{\mathcal{P}_0} |g| g^{rs} \partial_s q_a \right) = 0. \tag{15}$$

Using the relation $\dot{\vec{x}}^2 = \sum_{a=1}^{p} q_a^2 \dot{\theta}_a^2$ one can present the energy density $\mathcal{P}_0$ (5) as a function of the velocity components $\dot{\theta}_a(t)$

$$\mathcal{P}_0 = T \sqrt{\frac{|g|}{1 - \sum_{a=1}^{p} q_a^2 \dot{\theta}_a^2}}. \tag{16}$$

Taking into account the energy density conservation $\dot{\mathcal{P}}_0 = 0$ together with the time-independence of the metric $g_{rs} = \mathbf{q}_{,r} \mathbf{q}_{,s}$ (13) we obtain the solution for the polar angles $\theta_a(t)$

$$\theta_a(t) = \theta_{0a} + \omega_a t, \tag{17}$$

where θ_{0a} and ω_a are the integration constants. Then $\mathcal{P}_0$ (16) asquires the form

$$\mathcal{P}_0(\sigma^r) = T \sqrt{\frac{|g|}{1 - \sum_{a=1}^{p} \omega_a^2 q_a^2}} \tag{18}$$

which shows that the coordinates θ_a are cyclic and their conjugate momenta $j_a := \frac{\partial \mathcal{L}}{\partial \dot{\theta}_a} = \vec{\mathcal{P}} \frac{\partial \dot{\vec{x}}}{\partial \dot{\theta}_a}$ are the integrals of motion

$$\frac{dj_a}{dt} = 0, \quad j_a = \mathcal{P}_0 q_a^2 \dot{\theta}_a \equiv \mathcal{P}_0 \omega_a q_a^2, \quad (a = 1, 2, .., p) \tag{19}$$

additional to the Hamiltonian density $\mathcal{H}_0$ (4) taking the form

$$\mathcal{H}_0 = \sqrt{\sum_{a=1}^{p} (j_a/q_a)^2 + T^2 |g|}. \tag{20}$$

The functions j_a have the physical sense of the components of the angular momentum density associated with the generators of rotations in the planes x_1x_2, x_3x_4 ,...,$x_{2p-1}x_{2p}$ which form the above discussed abelian group $U(1)^p$

$$j_b = T \omega_b q_b^2 \sqrt{\frac{|g|}{1 - \sum_{a=1}^{p} \omega_a^2 q_a^2}}. \tag{21}$$

To solve Eqs. (15) we fix the gauge for the residual gauge symmetry (9)

$$q_1(\sigma^r) = q_1(\sigma^1), \quad q_2(\sigma^r) = q_2(\sigma^2), \ ...., \ q_p(\sigma^r) = q_p(\sigma^p) \tag{22}$$

that diagonalizes the derivatives $q_{a,s} = \delta_{as} \acute{q}_s$, where $\acute{q}_s := \frac{dq_s}{d\sigma^s}$, and g_{rs}

$$g_{rs} = \delta_{rs} \acute{q}_s^2, \quad g^{rs} = \frac{\delta_{rs}}{\acute{q}_s^2}, \quad g = \prod_{a=1}^{p} \acute{q}_a^2 \equiv \prod \acute{q}_a^2. \tag{23}$$

Using the representation $gg^{rs} = \frac{\delta_{rs}}{\acute{q}_r^2} \prod \acute{q}_a^2$ one can present Eqs. (15) as

$$\omega_r^2 q_r + \frac{1}{2\acute{q}_r} \partial_r \left(\frac{T}{\mathcal{P}_0}\right)^2 \prod \acute{q}_a^2 + \left(\frac{T}{\mathcal{P}_0}\right)^2 \frac{q_r''}{\acute{q}_r^2} \prod \acute{q}_a^2 = 0. \tag{24}$$

Then taking into account the representation

$$\left(\frac{T}{\mathcal{P}_0}\right)^2 = \frac{1 - \sum_{a=1}^{p} \omega_a^2 q_a^2}{\prod \acute{q}_a^2} \tag{25}$$

and calculating its partial derivatives in σ^r

$$\frac{1}{2} \partial_r \left(\frac{T}{\mathcal{P}_0}\right)^2 = -\omega_r^2 q_r \frac{\acute{q}_r}{\prod \acute{q}_a^2} - \left(\frac{T}{\mathcal{P}_0}\right)^2 \frac{q_r''}{\acute{q}_r} \tag{26}$$

we observe that the additives composing the second term in (24) are mutually cancelled with its first and third terms. This proves that anzats (12) written down in the gauge (22), containing p arbitrary periodic functions of one variable $q_a(\sigma^a) = q_a(\sigma^a + 2\pi)$,

$$\vec{x}^T = (q_1(\sigma^1)\cos\omega_1 t, q_1(\sigma^1)\sin\omega_1 t, \dots, q_p(\sigma^p)\cos\omega_p t, q_p(\sigma^p)\sin\omega_p t) \quad (27)$$

is an exact solution of the spinning p-brane equations with the fixed initial data $\theta_{0a} = 0$ at $t = 0$. This solution describes flat toroidal p-branes.

The periodicity conditions for q_a do not permit to consider p-dimensional hyperplanes as alternative flat p-brane hypersurfaces. However, one can consider p-branes with boundaries and to suppose that $\sigma^r \in [0, \infty)$. Then it is possible to choose the gauge (22) with the linear functions $q_s(\sigma^s)$

$$q_1(\sigma^r) = l_1 + k_1\sigma^1, \quad q_2(\sigma^r) = l_2 + k_2\sigma^2, \ \dots. \ , q_p(\sigma^r) = l_p + k_p\sigma^p, \qquad (28)$$

where l_a, k_a are arbitrary constants with the dimension of length.

As a result, the anzats (27) added by (28) turns out to describe spinning p-dimensional hyperplanes embedded in the $2p$-dimensional Euclidean space.

5 Summary

New exact solutions describing $U(1)^p$-invariant spinning p-branes embedded in the $D = (2p + 1)$-dimensional Minkowski space are constructed. Solutions for the compact hypersurfaces Σ_p of closed p-branes (with $p = 2, 3, ..., (D-1)/2$) are shown to be isometric to flat p-dimensional tori with time-independent radii. The solutions for the noncompact hypersurfaces of $U(1)^p$-invariant p-branes without boundaries are found to be presented by p-dimensional hyperplanes. The Hamiltonians desribing these spinning p-branes are presented.

The found exact solutions, in particular, for $p = 5$ present another exact solution for the 5-brane of $D = 11$ M/string theory additional to the solution describing the p-brane collapse [22]. The compact solutions may be generalized, in particular, to the Minkowski spaces with toroidal compactifications.

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